

ON HYPERBOLICITY OF 13-MOMENT SYSTEM

ZHENNING CAI

CAPT & School of Mathematical Sciences
Peking University
Yiheyuan Road 5, 100871 Beijing, China

YUWEI FAN

School of Mathematical Sciences
Peking University
Yiheyuan Road 5, 100871 Beijing, China

RUO LI

CAPT & School of Mathematical Sciences
Peking University
Yiheyuan Road 5, 100871 Beijing, China

(Communicated by Yan Guo)

ABSTRACT. We point out that the thermodynamic equilibrium is not an interior point of the hyperbolicity region of Grad's 13-moment system. With a compact expansion of the phase density, which is compacter than Grad's expansion, we derived a modified 13-moment system. The new 13-moment system admits the thermodynamic equilibrium as an interior point of its hyperbolicity region. We deduce a concise criterion to ensure the hyperbolicity, thus the hyperbolicity region can be quantitatively depicted.

1. Introduction. Grad's 13-moment system [4] has been studied for over 50 years. This system was derived by utilizing the isotropic Hermite expansion [3] of the phase density in the Boltzmann equation, and such an idea opened a brand new direction in the gas kinetic theory. In the subsequent years, a number of defects of this model were discovered, one of which was that Grad's 13-moment system is not globally hyperbolic, and for 1D flows, the hyperbolicity can only be obtained near the equilibrium [9]. The loss of hyperbolicity directly breaks the well-posedness of the partial differential equations, and thus the capability of this model is strictly limited. Extensions of this model include the regularized Burnett equations [6], regularized 13-moment equations [12, 11], and the Pearson-13-moment equations [14]. These methods may alleviate the problem of hyperbolicity to some extent [5, 13, 14]. However, all the analysis on the hyperbolicity is restricted to the 1D flows, while there are no comments indicating that the 3D case is the same or similar as the 1D case. Actually, Grad's paper [4] has pointed out that in comparison with the 1D flows, an additional soundspeed appears in the 2D flows. In this paper, we are concerned with the hyperbolicity for the full 3D flows.

2010 *Mathematics Subject Classification.* 82C40, 35L60.

Key words and phrases. Grad's moment system, hyperbolicity, modified 13-moment system.

The research is supported in part by the National Basic Research Program of China (2011CB309704).

We first point out that in the 3D Grad's 13-moment equations, for each equilibrium state, none of its neighbourhoods is contained in the hyperbolicity region. This reveals that the manifold formed by all the equilibrium states is on the boundary of the hyperbolicity region, and thus an arbitrary small perturbation of the equilibrium state may lead to the loss of hyperbolicity, which indicates the instability of Grad's 13-moment equations. More precisely, we prove that if an arbitrary small anisotropic perturbation is applied to the phase density from the equilibrium, the hyperbolicity may be broken down. Noticing that the anisotropy plays an essential role in breaking down the hyperbolicity, we then propose a new modified 13-moment model such that the equilibrium state lies in the interior of the hyperbolicity region. This modified system is derived by an anisotropic Hermite expansion instead of the isotropic Hermite expansion in Grad's method, where the anisotropy is specified by the temperature tensor. We find out a dimensionless quantity which can prescribe the departure of the phase density from the equilibrium state. It is proven that if this quantity is controlled above by a threshold, the full 3D system is hyperbolic. The value of this threshold is approximately given, and the size of the hyperbolicity region is depicted using the Chapman-Enskog type expansion.

The rest of this paper is as follows: in Section 2, some algebraic lemmas are given as preliminaries; in Section 3, the hyperbolicity of Grad's 13-moment equations is discussed for both 1D and 3D flows; Section 4 gives a modified 13-moment system, and its hyperbolicity is discussed in detail; finally, some concluding remarks are given in 5.

2. Some preliminary results in linear algebra. In this paper, we will focus on the hyperbolicity of moment systems, i.e. the real diagonalizability of the coefficient matrices in these systems. In order to make our later derivation self-consistent, we present some lemmas about matrices and polynomials as preliminaries and most of the proofs can be found in textbooks of linear algebra. If not specified, we are considering real matrices and polynomials with real coefficients.

Lemma 2.1 (Cayley-Hamilton). *For any square matrix \mathbf{A} and its characteristic polynomial $p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$, we have $p(\mathbf{A}) = \mathbf{0}$.*

Lemma 2.2. *For a square matrix \mathbf{A} , the following three statements are equivalent:*

1. λ is a root of the minimal polynomial of \mathbf{A} ,
2. λ is a root of the characteristic polynomial of \mathbf{A} ,
3. λ is an eigenvalue of \mathbf{A} .

Lemma 2.3. *A square matrix \mathbf{A} is diagonalizable if and only if the minimal polynomial of \mathbf{A} is the product of distinct linear functions.*

The following corollary can be derived from the above results:

Corollary 1. *For a square matrix \mathbf{A} , suppose $p(\lambda)$ is its characteristic polynomial. If there exists a polynomial $q(\lambda)$ such that $p(\lambda)$ and $q(\lambda)$ share the same roots, and $q(\mathbf{A}) \neq \mathbf{0}$, then \mathbf{A} is not diagonalizable.*

Proof. Let us prove it by contradiction. We suppose \mathbf{A} is diagonalizable. According to Lemma 2.2 and Lemma 2.3, the minimal polynomial of \mathbf{A} can be written as

$$m(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n), \quad (1)$$

where $\lambda_1, \dots, \lambda_n$ are all the distinct eigenvalues of \mathbf{A} . Since $p(\lambda)$ and $q(\lambda)$ share the same roots, we have $m(\lambda) \mid q(\lambda)$, and thus $q(\mathbf{A}) = \mathbf{0}$, which violates the condition $q(\mathbf{A}) \neq \mathbf{0}$. \square

We will use Lemma 2.3 when proving a matrix is diagonalizable, and use Corollary 1 when proving a matrix is not diagonalizable. The following lemma can be used to tell if a polynomial has multiple roots.

Lemma 2.4. *Let $p(z)$ be a polynomial defined on complex numbers. Then $p(z)$ has no multiple roots if and only if $p(z)$ and $\frac{d}{dz}p(z)$ are coprime.*

Definition 2.5. Let $p(z)$ and $q(z)$ be two polynomials defined as

$$p(z) = p_0 + p_1z + p_2z^2 + \cdots + p_mz^m, \quad q(z) = q_0 + q_1z + q_2z^2 + \cdots + q_nz^n. \quad (2)$$

The $(m+n) \times (m+n)$ matrix

$$\text{Syl}(p, q) := \begin{pmatrix} p_m & \cdots & \cdots & p_0 & & & \\ & \ddots & & & \ddots & & \\ & & p_m & \cdots & \cdots & p_0 & \\ q_n & \cdots & \cdots & q_0 & & & \\ & \ddots & & & \ddots & & \\ & & q_n & \cdots & \cdots & q_0 & \end{pmatrix} \begin{matrix} \left. \vphantom{\begin{pmatrix} p_m & \cdots & \cdots & p_0 \\ & \ddots & & \\ & & p_m & \cdots & \cdots & p_0 \end{pmatrix}} \right\} n \text{ rows} \\ \left. \vphantom{\begin{pmatrix} q_n & \cdots & \cdots & q_0 \\ & \ddots & & \\ & & q_n & \cdots & \cdots & q_0 \end{pmatrix}} \right\} m \text{ rows} \end{matrix} \quad (3)$$

is called the Sylvester matrix associated to $p(z)$ and $q(z)$. The resultant of $p(z)$ and $q(z)$ is defined as the determinant of the above matrix:

$$\text{res}(p, q) := \det(\text{Syl}(p, q)). \quad (4)$$

The following lemma is to be used to tell if two polynomials have common roots.

Lemma 2.6. *Two nonzero polynomials are coprime if and only if their resultant is nonzero.*

3. Boltzmann equation and Grad's 13-moment system. From this section, we start our discussion on the kinetic models. Section 3.1 and 3.2 give brief introductions to the Boltzmann equation and Grad's 13-moment system respectively, and Section 3.3 reviews the classical knowledge on the hyperbolicity of Grad's 13-moment system for 1D flows. In the last part of this section, we present our new findings that an intrinsic difference exists between the hyperbolicity regions in the 1D and 3D cases for Grad's system.

3.1. Boltzmann equation. The Boltzmann equation is a fundamental physical model in the gas kinetic theory. Suppose $f(t, \mathbf{x}, \boldsymbol{\xi})$ is the function of phase density, where t is the time, $\mathbf{x} = (x_1, x_2, x_3)^T$ is the spatial coordinates, and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T$ stands for the velocity of gas molecules. Then the Boltzmann equation reads

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} f = Q(f, f). \quad (5)$$

The right hand side $Q(f, f)$ describes the interaction between particles:

$$Q(f, f) = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi/2} (f'_* f' - f_* f) |\boldsymbol{\xi} - \boldsymbol{\xi}_*| \sigma \sin \Theta \, d\Theta \, d\varepsilon \, d\boldsymbol{\xi}_*, \quad (6)$$

where $f' = f(t, \mathbf{x}, \boldsymbol{\xi}')$, $f_* = f(t, \mathbf{x}, \boldsymbol{\xi}_*)$, $f'_* = f(t, \mathbf{x}, \boldsymbol{\xi}'_*)$, and the velocities $\boldsymbol{\xi}$, $\boldsymbol{\xi}_*$ and $\boldsymbol{\xi}'$, $\boldsymbol{\xi}'_*$ are the pre- and post-collision velocities of a colliding pair of molecules, and σ is the differential cross-section. In this paper, we consider only the Maxwell molecules, for which $|\boldsymbol{\xi} - \boldsymbol{\xi}_*| \sigma$ is a function of Θ . We refer the readers to [2, 1] for more details of the collision term and the Maxwell molecules.

The gas kinetic theory describes the fluid states in a microscopic view, while the macroscopic quantities such as density, velocity and temperature can be obtained by integrations. Define

$$\langle h \rangle = m \int_{\mathbb{R}^3} h \, d\boldsymbol{\xi}, \quad (7)$$

where m is the mass of a single gas molecule. Then the relations between the density function f and some common macroscopic quantities are as follows:

- Density: $\rho = \langle f \rangle$;
- Velocity: $\mathbf{u} = (u_1, u_2, u_3)^T = \rho^{-1} \langle \boldsymbol{\xi} f \rangle$;
- Temperature: $T = (3\rho k_B/m)^{-1} \langle |\boldsymbol{\xi} - \mathbf{u}|^2 f \rangle$, where k_B is the Boltzmann constant;
- Temperature tensor: $T_{ij} = (\rho k_B/m)^{-1} \langle (\xi_i - u_i)(\xi_j - u_j) f \rangle$, $i, j = 1, 2, 3$;
- Heat flux: $\mathbf{q} = (q_1, q_2, q_3)^T = \langle |\boldsymbol{\xi} - \mathbf{u}|^2 (\boldsymbol{\xi} - \mathbf{u}) f / 2 \rangle$.

Following the conventional style, we denote

$$\theta = \frac{k_B}{m} T, \quad \theta_{ij} = \frac{k_B}{m} T_{ij}, \quad \boldsymbol{\Theta} = (\theta_{ij})_{3 \times 3}. \quad (8)$$

It can be derived from the positivity of the density function f that $\boldsymbol{\Theta}$ is symmetric positive definite. For simplicity, below we denote the relative velocity $\boldsymbol{\xi} - \mathbf{u}$ by \mathbf{C} , and the norm of \mathbf{C} is denoted by C . For example, we have

$$\theta = (3\rho)^{-1} \langle C^2 f \rangle, \quad \theta_{ij} = \rho^{-1} \langle C_i C_j f \rangle. \quad (9)$$

3.2. Grad's 13-moment system. The high dimensionality of the Boltzmann equation introduces extreme difficulties to its numerical treatment. In order to simplify the model, Grad proposed a 13-moment system [4], in which the velocity variable $\boldsymbol{\xi}$ was eliminated, while only 13 equations are presented. These equations are derived by assuming the following particular form of the phase density f :

$$f|_{13} = \left[1 + \frac{\theta_{ij} - \delta_{ij}\theta}{2\theta^2} (C_i C_j - \delta_{ij} C^2) + \frac{2}{5} \frac{q_k}{\rho\theta^2} C_k \left(\frac{C^2}{2\theta} - \frac{5}{2} \right) \right] f_M, \quad (10)$$

where f_M is the Maxwellian, defined as

$$f_M = \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{C^2}{2\theta}\right), \quad (11)$$

and in (10), the Einstein summation convention is assumed. Accordingly, when an index appears twice in a single term, it implies summation of that term over all the values of the index. By (10), Grad's 13-moment system can be written as a closed system as

$$\left\langle \phi \frac{\partial f|_{13}}{\partial t} \right\rangle + \langle \phi (\boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} f|_{13}) \rangle = \langle \phi Q(f|_{13}, f|_{13}) \rangle, \quad (12)$$

where

$$\begin{aligned} \phi = & (1, \\ & C_1, C_2, C_3, \\ & C_1^2, C_2^2, C_3^2, C_1 C_2, C_1 C_3, C_2 C_3, \\ & C^2 C_1, C^2 C_2, C^2 C_3)^T. \end{aligned}$$

By simplifications of the expression obtained after the integrations, the above system can be explicitly given by

$$\begin{aligned}
\frac{d\rho}{dt} + \rho \frac{\partial u_k}{\partial x_k} &= 0, \\
\frac{du_i}{dt} + \frac{\theta_{ik}}{\rho} \frac{\partial \rho}{\partial x_k} + \frac{\partial \theta_{ik}}{\partial x_k} &= 0, \quad i = 1, 2, 3, \\
\frac{d\theta_{ij}}{dt} + 2\theta_{k(i} \frac{\partial u_{j)}}{\partial x_k} + \frac{1}{\rho} \left(\frac{4}{5} \frac{\partial q_{(i}}{\partial x_j)} + \frac{2}{5} \delta_{ij} \frac{\partial q_k}{\partial x_k} \right) &= -\frac{\rho\theta}{\mu} (\theta_{ij} - \delta_{ij}\theta), \quad i, j = 1, 2, 3, \\
\frac{dq_i}{dt} - (\theta_{ij}\theta_{jk} - 2\theta\theta_{ik} + \theta^2\delta_{ik}) \frac{\partial \rho}{\partial x_k} + \frac{7}{5} q_i \frac{\partial u_k}{\partial x_k} + \frac{7}{5} q_k \frac{\partial u_i}{\partial x_k} + \frac{2}{5} q_k \frac{\partial u_k}{\partial x_i} \\
&\quad - \rho\theta_{ik} \left(\frac{\partial \theta_{jk}}{\partial x_j} - \frac{7}{6} \frac{\partial \theta_{jj}}{\partial x_k} \right) + 2\rho\theta \left(\frac{\partial \theta_{ik}}{\partial x_k} - \frac{1}{3} \frac{\partial \theta_{jj}}{\partial x_i} \right) = -\frac{2\rho\theta}{3\mu} q_i, \quad i = 1, 2, 3.
\end{aligned} \tag{13}$$

Here,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}$$

is the material derivative, and the brackets around indices denote the symmetrization of a tensor. The symbol μ denotes the coefficient of viscosity. For Maxwell molecules, μ is proportional to θ .

3.3. Local hyperbolicity of 1D Grad's moment system. It is well-known that the 1D Grad's moment system is hyperbolic only when the fluid is near the thermodynamic equilibrium state. For 1D flows, the 13-moment system reduces to a smaller system containing only five equations, which are obtained by setting $u_2 = u_3 = \theta_{12} = \theta_{13} = \theta_{23} = q_2 = q_3 = 0$ and $\theta_{33} = \theta_{22}$ in (13). Such operation eliminates eight of the thirteen variables, and results in the 1D system

$$\frac{\partial \hat{\mathbf{w}}}{\partial t} + \hat{\mathbf{M}}(\hat{\mathbf{w}}) \frac{\partial \hat{\mathbf{w}}}{\partial x} = \hat{\mathbf{Q}}(\hat{\mathbf{w}}), \tag{14}$$

where $\hat{\mathbf{w}} = (\rho, u_1, \theta_{11}, \theta_{22}, q_1)^T$, $\hat{\mathbf{Q}}(\hat{\mathbf{w}}) = (0, 0, \rho\theta(\theta - \theta_{11})/\mu, \rho\theta(\theta - \theta_{22})/\mu, -\frac{2}{3}\rho\theta q_1/\mu)^T$, and

$$\hat{\mathbf{M}}(\hat{\mathbf{w}}) = \begin{pmatrix} u_1 & \rho & 0 & 0 & 0 \\ \theta_{11}/\rho & u_1 & 1 & 0 & 0 \\ 0 & 2\theta_{11} & u_1 & 0 & 6/(5\rho) \\ 0 & 0 & 0 & u_1 & 2/(5\rho) \\ -4(\theta_{11} - \theta_{22})^2/9 & 16q_1/5 & \rho(11\theta_{11} + 16\theta_{22})/18 & \rho(17\theta_{11} - 8\theta_{22})/9 & u_1 \end{pmatrix}. \tag{15}$$

The system (14) is hyperbolic if and only if $\hat{\mathbf{M}}(\hat{\mathbf{w}})$ is real diagonalizable.

In order to check the diagonalizability of $\hat{\mathbf{M}}(\hat{\mathbf{w}})$, we calculate its characteristic polynomial as

$$\begin{aligned}
\det(\lambda \mathbf{I} - \hat{\mathbf{M}}) &= (\lambda - u_1) \left[(\lambda - u_1)^4 - \frac{2}{45} (101\theta_{11} + 16\theta_{22})(\lambda - u_1)^2 \right. \\
&\quad \left. - \frac{96}{25} \frac{q_1}{\rho} (\lambda - u_1) + \frac{1}{15} (53\theta_{11}^2 - 16\theta_{11}\theta_{22} + 8\theta_{22}^2) \right].
\end{aligned} \tag{16}$$

We introduce the dimensionless quantity $\hat{\lambda} = (\lambda - u_1)/\sqrt{\theta}$, and then the equation $\det(\lambda \mathbf{I} - \hat{\mathbf{M}}) = 0$ becomes

$$\hat{\lambda} \left[\hat{\lambda}^4 - \frac{2}{45} \frac{101\theta_{11} + 16\theta_{22}}{\theta} \hat{\lambda}^2 - \frac{96}{25} \frac{q_1}{\rho\theta^{3/2}} \hat{\lambda} + \frac{1}{15} \frac{53\theta_{11}^2 - 16\theta_{11}\theta_{22} + 8\theta_{22}^2}{\theta^2} \right] = 0. \quad (17)$$

Consider the special case $\theta_{11} = \theta_{22} = \theta$ and $q_1 = 0$, which implies the fluid is in its local equilibrium, all solutions of the above equation are

$$\hat{\lambda}_{1,5} = \pm \sqrt{\frac{13 + \sqrt{94}}{5}}, \quad \hat{\lambda}_{2,4} = \pm \sqrt{\frac{13 - \sqrt{94}}{5}}, \quad \hat{\lambda}_3 = 0. \quad (18)$$

Therefore, in this case, $\hat{\mathbf{M}}(\hat{\mathbf{w}})$ has no multiple eigenvalues, thus is real diagonalizable. If $(\theta_{11} - \theta_{22})/\theta$ and $q_1/(\rho\theta^{3/2})$ are small enough, the roots of (17) are small perturbations of (18), which are still real and separable. This shows that there is a hyperbolicity region for 1D moment system around the thermodynamic equilibrium, and the Maxwell distribution is an interior point of the hyperbolicity region. A precise depiction of the hyperbolicity region can be found in [9].

3.4. Lack of hyperbolicity of 3D Grad's 13-moment system. To the best of our knowledge, there has not been any published investigation on the hyperbolicity of the full 3D Grad's system. One may take it for granted that the full 3D case is similar as the 1D case and there exists a neighbourhood of the equilibrium such that the system is hyperbolic. Unfortunately, this is not true. In this section, we are going to show that Maxwellian is on the boundary of the hyperbolicity region. The analysis below contains some tedious calculations, which are carried out by the computer algebra system Mathematica [10].

In the 3D case, Grad's 13-moment equations can also be written in the quasi-linear form as

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{M}_k(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k} = \mathbf{Q}(\mathbf{w}). \quad (19)$$

Now \mathbf{w} is a vector with 13 entries:

$$\mathbf{w} = (\rho, u_1, u_2, u_3, \theta_{11}, \theta_{22}, \theta_{33}, \theta_{12}, \theta_{13}, \theta_{23}, q_1, q_2, q_3)^T.$$

The expressions of the matrices \mathbf{M}_k and the operator \mathbf{Q} can be obtained from (13). Since Grad's moment system is rotationally invariant, in order to check the hyperbolicity of (19), we only need to check the diagonalizability of \mathbf{M}_1 . As a reference, the precise form of $\mathbf{M}_1(\mathbf{w})$ is given on page 8.

When \mathbf{w} represents the equilibrium state, which means

$$\theta_{12} = \theta_{13} = \theta_{23} = q_1 = q_2 = q_3 = 0, \quad \theta_{11} = \theta_{22} = \theta_{33} = \theta. \quad (20)$$

The characteristic polynomial of $\mathbf{M}_1(\mathbf{w})$ is

$$\det(\lambda \mathbf{I} - \mathbf{M}_1) = \frac{1}{125} (\lambda - u_1)^5 [5(\lambda - u_1)^2 - 7\theta]^2 [5(\lambda - u_1)^4 - 26\theta(\lambda - u_1)^2 + 15\theta^2].$$

All roots of the above polynomial are

$$u_1, \quad u_1 \pm \sqrt{\frac{7}{5}}\theta, \quad u_1 \pm \sqrt{\frac{13 + \sqrt{94}}{5}}\theta, \quad u_1 \pm \sqrt{\frac{13 - \sqrt{94}}{5}}\theta.$$

Thus the eigenvalues of \mathbf{M}_1 are all real. In order to check its diagonalizability, let

$$q(\lambda) = \frac{1}{25} (\lambda - u_1) [5(\lambda - u_1)^2 - 7\theta] \cdot [5(\lambda - u_1)^4 - 26\theta(\lambda - u_1)^2 + 15\theta^2].$$

Direct verification shows $q(\mathbf{M}_1) = \mathbf{0}$. According to Lemma 2.3, $\mathbf{M}_1(\mathbf{w})$ is real diagonalizable at the equilibrium state.

In order to show that the equilibrium is on the boundary of the hyperbolicity region, we consider the following case:

$$\theta_{13} = \theta_{23} = q_1 = q_2 = q_3 = 0, \quad \theta_{11} = \theta_{22} = \theta_{33} = \theta. \quad (21)$$

When f is the following Gaussian distribution:

$$f = \frac{\rho}{\sqrt{\det(2\pi\Theta)}} \exp\left(-\frac{1}{2}\mathbf{C}^T\Theta^{-1}\mathbf{C}\right), \quad \Theta = \begin{pmatrix} \theta & \theta_{12} & 0 \\ \theta_{12} & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}, \quad (22)$$

the relation (21) is satisfied. When $|\theta_{12}| < \theta$, the matrix Θ is positive definite, and thus the distribution function (22) can be a physical configuration. Substituting (21) into (3.4) and calculating the characteristic polynomial of $\mathbf{M}_1(\mathbf{w})$, one has

$$\det(\lambda\mathbf{I} - \mathbf{M}_1) = \frac{1}{125}(\lambda - u_1)^3[5(\lambda - u_1)^2 - 7\theta] \cdot r\left(\frac{(\lambda - u_1)^2}{\theta}\right),$$

where

$$r(x) = 25x^4 - 165x^3 + \left(257 + 48\frac{\theta_{12}^2}{\theta^2}\right)x^2 + \left(8\frac{\theta_{12}^2}{\theta^2} - 105\right)x - 28\frac{\theta_{12}^2}{\theta^2}.$$

Let

$$q(\lambda) = (\lambda - u_1)[5(\lambda - u_1)^2 - 7\theta] \cdot r\left(\frac{(\lambda - u_1)^2}{\theta}\right).$$

Obviously $q(\lambda)$ and $\det(\lambda\mathbf{I} - \mathbf{M}_1)$ share the same roots. Direct calculation of $q(\mathbf{M}_1)$ gives us that

$$q(\mathbf{M}_1) = \frac{56\theta^2\theta_{12}^3}{\rho}(\rho\theta\mathbf{E}_{10,4} - \mathbf{E}_{10,13}),$$

where $\mathbf{E}_{i,j} = \mathbf{e}_i\mathbf{e}_j^T$, and \mathbf{e}_j is the unit vector with the j -th entry being 1. According to Corollary 1, if $\theta_{12} \neq 0$, then $\mathbf{M}_1(\mathbf{w})$ is not diagonalizable. Actually, one may find that $r(x)$ have at least one negative root since $r(-\infty) > 0$ and $r(0) < 0$, and therefore $\mathbf{M}_1(\mathbf{w})$ has eigenvalues with nonzero imaginary parts, which also violates the hyperbolic condition.

The above analysis shows that when (21) and $\theta_{12} \neq 0$ holds, the hyperbolicity of (19) breaks down, no matter how small the value of θ_{12} is. It turns out that there does not exist a neighbourhood of the equilibrium such that all the states in this neighbourhood lead to the hyperbolicity of Grad's 13-moment system. Without the hyperbolicity in a neighbourhood of the equilibrium, the wellposedness of the Grad's 13-moment system is not guaranteed even if the phase density is extremely close to the equilibrium. This severe drawback may be the possible reason why there are hardly any positive evidences for the Grad's 13-moment system in the last decades.

$$\mathbf{M}_1(\mathbf{w}) = \begin{pmatrix} u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\theta_{11}}{\rho} & u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\theta_{12}}{\rho} & 0 & u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\theta_{13}}{\rho} & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2\theta_{11} & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{5\rho} & 0 & 0 \\ 0 & 0 & 2\theta_{12} & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 0 & \frac{2}{5\rho} & 0 & 0 \\ 0 & 0 & 0 & 2\theta_{13} & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & \frac{2}{5\rho} & 0 & 0 \\ 0 & \theta_{12} & \theta_{11} & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & \frac{2}{5\rho} & 0 \\ 0 & \theta_{13} & 0 & \theta_{11} & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & \frac{2}{5\rho} \\ 0 & 0 & \theta_{13} & \theta_{12} & 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 \\ -(\theta - \theta_{11})^2 - (\theta_{12}^2 + \theta_{13}^2) & \frac{16q_1}{5} & \frac{2q_2}{5} & \frac{2q_3}{5} & \frac{\rho(\theta_{11}+8\theta)}{6} & \frac{\rho(7\theta_{11}-4\theta)}{6} & \frac{\rho(7\theta_{11}-4\theta)}{6} & -\rho\theta_{12} & -\rho\theta_{13} & 0 & u_1 & 0 & 0 & 0 \\ \theta_{12}\theta_{33} - \theta_{13}\theta_{23} - \theta\theta_{12} & \frac{7q_2}{5} & \frac{7q_1}{5} & 0 & \frac{\rho\theta_{12}}{6} & \frac{7\rho\theta_{12}}{6} & \frac{7\rho\theta_{12}}{6} & \rho(2\theta - \theta_{22}) & -\rho\theta_{23} & 0 & 0 & u_1 & 0 & 0 \\ \theta_{13}\theta_{22} - \theta_{12}\theta_{23} - \theta\theta_{13} & \frac{7q_3}{5} & 0 & \frac{7q_1}{5} & \frac{\rho\theta_{13}}{6} & \frac{7\rho\theta_{13}}{6} & \frac{7\rho\theta_{13}}{6} & -\rho\theta_{23} & \rho(2\theta - \theta_{33}) & 0 & 0 & 0 & u_1 & 0 \end{pmatrix}$$

4. Modified 13-moment System. The results in Section 3.4 reveal a crucial issue of Grad's original system. In order to establish the local hyperbolicity around the equilibrium state, we derive a modified 13-moment system in this section, which is hyperbolic for any states close enough to the equilibrium. The proofs will be given in detail, and the size of the hyperbolicity region will be discussed.

4.1. Derivation of the modified system. The modified 13-moment system is based on the following assumption of the phase density:

$$\tilde{f}|_{13} = \left[1 + \frac{2}{5\rho} \mathbf{s}^T \boldsymbol{\Theta}^{-1} \mathbf{C} \left(\frac{1}{2} \mathbf{C}^T \boldsymbol{\Theta}^{-1} \mathbf{C} - \frac{5}{2} \right) \right] f_G, \quad (23)$$

where $\mathbf{s} = (s_1, s_2, s_3)^T$, and f_G is a Gaussian distribution:

$$f_G = \frac{\rho}{m \sqrt{\det(2\pi \boldsymbol{\Theta})}} \exp \left(-\frac{1}{2} \mathbf{C}^T \boldsymbol{\Theta}^{-1} \mathbf{C} \right). \quad (24)$$

Comparing with f_M , the function f_G incorporates the whole temperature tensor into the exponent, and thus it can be expected that such an approximation includes more nonlinearity than (10), and is more suitable for describing anisotropic density functions. In order to meet the requirement of orthogonality, the vector \mathbf{s} should be related to the density function by

$$\mathbf{s} = \frac{1}{2} \langle C_G^2 \mathbf{C} f \rangle,$$

where $C_G^2 = \mathbf{C}^T \boldsymbol{\Theta}^{-1} \mathbf{C}$. For the postulate (23), the relation between \mathbf{s} and the heat flux \mathbf{q} is

$$q_j = \frac{3}{5} s_i \theta_{ij}.$$

Similar as the derivation of Grad's 13-moment system, the new moment system can be written as

$$\left\langle \tilde{\phi} \frac{\partial \tilde{f}|_{13}}{\partial t} \right\rangle + \left\langle \tilde{\phi} (\boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} \tilde{f}|_{13}) \right\rangle = \left\langle \tilde{\phi} Q(\tilde{f}|_{13}, \tilde{f}|_{13}) \right\rangle,$$

where

$$\begin{aligned} \tilde{\phi} = & (1, \\ & C_1, C_2, C_3, \\ & C_1^2, C_2^2, C_3^2, C_1 C_2, C_1 C_3, C_2 C_3, \\ & C_G^2 C_1, C_G^2 C_2, C_G^2 C_3)^T. \end{aligned}$$

We reformulate the resulting system in explicit form as

$$\frac{d\rho}{dt} + \rho \frac{\partial u_k}{\partial x_k} = 0, \quad (25a)$$

$$\frac{du_i}{dt} + \frac{\theta_{ik}}{\rho} \frac{\partial \rho}{\partial x_k} + \frac{\partial \theta_{ik}}{\partial x_k} = 0, \quad i = 1, 2, 3, \quad (25b)$$

$$\frac{d\theta_{ij}}{dt} + 2\theta_{k(i} \frac{\partial u_{j)}}{\partial x_k} + \frac{6}{5\rho} \left(s_{(i} \frac{\partial \theta_{jk)}}{\partial x_k} + \theta_{(ij} \frac{\partial s_{k)}}{\partial x_k} \right) = -\frac{\rho\theta}{\mu} (\theta_{ij} - \delta_{ij}\theta), \quad i, j = 1, 2, 3, \quad (25c)$$

$$\begin{aligned} \frac{ds_j}{dt} - \frac{6}{5}\theta^{ik}\theta_{l(i} s_{j)} \frac{\partial u_k}{\partial x_l} - \frac{18}{25\rho}\theta^{ik} \left(s_{(i} s_{j)} \frac{\partial \theta_{kl}}{\partial x_l} + s_{(i}\theta_{jk} \frac{\partial s_{l)}}{\partial x_l} \right) \\ + \frac{1}{2} \left(\rho\theta^{ik}\theta_{jl} \frac{\partial \theta_{ik}}{\partial x_l} + 2\rho \frac{\partial \theta_{jl}}{\partial x_l} \right) + \frac{2}{5} \left(7s_{(j} \frac{\partial u_{l)}}{\partial x_l} + \theta^{ik}\theta_{jl} s_{(i} \frac{\partial u_k}{\partial x_l} \right) = \tilde{Q}_j, \quad j = 1, 2, 3. \end{aligned} \quad (25d)$$

In equation (25d), θ^{ij} stands for the (i, j) entry of matrix Θ^{-1} , and

$$\tilde{Q}_j = -\frac{\rho\theta}{\mu} \left(\frac{71}{30}s_j - \frac{9}{10}\theta\theta^{ii}s_j - \frac{1}{15}\theta^{ii}\theta_{jk}s_k \right).$$

The expressions of \tilde{Q}_j are obtained by using the following properties of Maxwell molecules:

$$\begin{aligned} \langle C_i C_j Q(f, f) \rangle &= -\frac{\rho\theta}{\mu} \left\langle \left(C_i C_j - \frac{1}{3}C^2\delta_{ij} \right) f \right\rangle, \quad \langle C^2 C_j Q(f, f) \rangle = -\frac{2}{3}\frac{\rho\theta}{\mu} \langle C^2 C_j f \rangle, \\ \left\langle \left(C_i C_j C_k - \frac{3}{5}C^2 C_{(i}\delta_{jk)} \right) Q(f, f) \right\rangle &= -\frac{3}{2}\frac{\rho\theta}{\mu} \left\langle \left(C_i C_j C_k - \frac{3}{5}C^2 C_{(i}\delta_{jk)} \right) f \right\rangle. \end{aligned}$$

The system (25) can also be written in a quasi-linear form:

$$\frac{\partial \tilde{\mathbf{w}}}{\partial t} + \tilde{\mathbf{M}}_k(\tilde{\mathbf{w}}) \frac{\partial \tilde{\mathbf{w}}}{\partial x_k} = \tilde{\mathbf{Q}}(\tilde{\mathbf{w}}), \quad (26)$$

and we choose

$$\tilde{\mathbf{w}} = (\rho, u_1, u_2, u_3, \theta_{11}, \theta_{22}, \theta_{33}, \theta_{12}, \theta_{13}, \theta_{23}, s_1, s_2, s_3)^T.$$

Since the linear space spanned by all the components of $\tilde{\boldsymbol{\phi}}$ is rotationally invariant, the moment equations (25) are also rotationally invariant. Therefore, below we focus on the first coefficient matrix $\tilde{\mathbf{M}}_1(\tilde{\mathbf{w}})$.

4.2. Local hyperbolicity of the modified system. Before establishing the local hyperbolicity of (25), we provide a technical lemma first:

Lemma 4.1. *For a given symmetric positive definite matrix $\Theta = (\theta_{ij})_{3 \times 3}$, the inequality*

$$\theta_{11}^{-1} s_1^2 \leq \mathbf{s}^T \Theta^{-1} \mathbf{s} \quad (27)$$

holds for any vector $\mathbf{s} = (s_1, s_2, s_3)^T \in \mathbb{R}^3$. The equality holds if and only if there exists a constant k such that

$$s_1 = k\theta_{11}, \quad s_2 = k\theta_{12}, \quad s_3 = k\theta_{13}. \quad (28)$$

Proof. We first prove $\theta_{12}\theta^{12} + \theta_{13}\theta^{13} \leq 0$. Let $\mathfrak{S} = \theta_{12}\theta^{12} + \theta_{13}\theta^{13}$. Then we have

$$\theta_{12}(\theta_{23}\theta_{13} - \theta_{12}\theta_{33}) + \theta_{13}(\theta_{12}\theta_{23} - \theta_{22}\theta_{13}) = \mathfrak{S}\det(\Theta). \quad (29)$$

This equation can be considered as a quadratic equation of θ_{13} , and its discriminant is

$$\Delta = (2\theta_{12}\theta_{23})^2 - \theta_{22}[4\theta_{12}^2\theta_{33} + \mathfrak{S}\det(\Theta)] = 4\theta_{12}^2(\theta_{23}^2 - \theta_{22}\theta_{33}) - \mathfrak{S}\theta_{22}\det(\Theta).$$

Since Θ is positive definite, the following inequalities hold:

$$\theta_{23}^2 - \theta_{22}\theta_{33} < 0, \quad \theta_{22} > 0, \quad \det(\Theta) > 0.$$

Thus, in order that (29) is not less than zero, $\mathfrak{S} \leq 0$ must hold. Moreover, if $\mathfrak{S} = 0$, θ_{12} must also be zero, and then (29) becomes $\theta_{22}\theta_{13}^2 = 0$, which means $\theta_{13} = 0$. Obviously when $\theta_{12} = \theta_{13} = 0$, one has $\mathfrak{S} = 0$. Therefore we finally conclude that $\mathfrak{S} \leq 0$, and the equality holds if and only if $\theta_{12} = \theta_{13} = 0$.

Now let $\mathfrak{D} = \mathbf{s}^T \Theta^{-1} \mathbf{s} - \theta_{11}^{-1} s_1^2$, which can be written as

$$(\theta_{11}\theta^{11} - 1)s_1^2 + 2\theta_{11}(\theta^{12}s_2 + \theta^{13}s_3)s_1 + \theta_{11}(\theta^{22}s_2^2 + 2\theta^{23}s_2s_3 + \theta^{33}s_3^2) = \theta_{11}\mathfrak{D}. \quad (30)$$

We consider the following two cases:

- If $\theta_{11}\theta^{11} - 1 = 0$, since $\theta_{1k}\theta^{1k} = 1$, one has $\theta_{12}\theta^{12} + \theta_{13}\theta^{13} = 0$. In this case, $\theta_{12} = \theta_{13} = 0$, and therefore

$$\theta^{12} = \frac{\theta_{13}\theta_{23} - \theta_{12}\theta_{33}}{\det(\Theta)} = 0, \quad \theta^{13} = \frac{\theta_{12}\theta_{23} - \theta_{13}\theta_{22}}{\det(\Theta)} = 0.$$

Thus (30) becomes $\theta^{22}s_2^2 + 2\theta^{23}s_2s_3 + \theta^{33}s_3^2 = \mathfrak{D}$, which is equivalent to $\tilde{\mathbf{s}}^T \Theta^{-1} \tilde{\mathbf{s}} = \mathfrak{D}$ for $\tilde{\mathbf{s}} = (0, s_2, s_3)^T$. Since Θ is positive definite, Θ^{-1} is also positive definite. This shows that $\mathfrak{D} \geq 0$, and the equality holds if and only if $s_2 = s_3 = 0$. In the case of $\mathfrak{D} = s_2 = s_3 = \theta_{12} = \theta_{13} = 0$, the relation (28) holds with $k = s_1/\theta_{11}$.

- If $\theta_{11}\theta^{11} - 1 \neq 0$, then $\theta_{11}\theta^{11} - 1 = -(\theta_{12}\theta^{12} + \theta_{13}\theta^{13}) > 0$. In this case, (30) is a quadratic equation of s_1 , whose discriminant is

$$\begin{aligned} \Delta &= [2\theta_{11}(\theta^{12}s_2 + \theta^{13}s_3)]^2 - 4\theta_{11}(\theta_{11}\theta^{11} - 1)(\theta^{22}s_2^2 + 2\theta^{23}s_2s_3 + \theta^{33}s_3^2 - \mathfrak{D}) \\ &= -4\theta_{11}(\theta_{12}s_3 - \theta_{13}s_2)^2/\det(\Theta) + 4\theta_{11}\mathfrak{D}(\theta_{11}\theta^{11} - 1). \end{aligned}$$

In order that s_1 is real, $\mathfrak{D} \geq 0$ must hold. And if $\mathfrak{D} = 0$, $\theta_{12}s_3 - \theta_{13}s_2$ must be zero. Thus when \mathfrak{D} is zero, there exist a constant k such that

$$s_2 = k\theta_{12}, \quad s_3 = k\theta_{13}. \quad (31)$$

Substitute (31) and $\mathfrak{D} = 0$ into (30), it can be solved that $s_1 = k\theta_{11}$.

In both cases, (27) holds, and it has been demonstrated that if $\theta_{11}^{-1}s_1^2 = \mathbf{s}^T \Theta^{-1} \mathbf{s}$, then (28) holds. It only remains to prove that $\theta_{11}^{-1}s_1^2 = \mathbf{s}^T \Theta^{-1} \mathbf{s}$ is a conclusion of (28).

If (28) holds, then

$$(k, 0, 0)\Theta = k(\theta_{11}, \theta_{12}, \theta_{13}) = \mathbf{s}^T.$$

Therefore $\mathbf{s}^T \Theta^{-1} = (k, 0, 0)$, and then

$$\mathbf{s}^T \Theta^{-1} \mathbf{s} = (k, 0, 0)\mathbf{s} = k^2\theta_{11} = \theta_{11}^{-1}s_1^2. \quad (32)$$

This completes the proof of the lemma. \square

Now we claim that the modified 13-moment system (25) is locally hyperbolic around the equilibrium. Precisely, we have the following major theorem of this section:

Theorem 4.2. *There exists a positive constant $\delta > 0$, such that if $\rho^{-2}\mathbf{s}^T \Theta^{-1} \mathbf{s} < \delta$, $\tilde{\mathbf{M}}_1(\tilde{\mathbf{w}})$ is real diagonalizable.*

Proof. Let

$$\eta_1 := \rho^{-2} \theta_{11}^{-1} s_1^2, \quad \eta_2 := \rho^{-2} \mathbf{s}^T \boldsymbol{\Theta}^{-1} \mathbf{s}, \quad \zeta = \frac{\lambda - u_1}{\sqrt{\theta_{11}}}.$$

According to Lemma 4.1, we have $\eta_1 \leq \eta_2 < \delta$. By direct calculation, the characteristic polynomial of $\tilde{\mathbf{M}}_1(\mathbf{w})$ is

$$p(\lambda) := \det(\lambda \mathbf{I} - \tilde{\mathbf{M}}_1) = \frac{(\sqrt{\theta_{11}})^{13}}{1953125} [p_{11}(\zeta) + \eta_2 p_{12}(\zeta)] [p_{21}(\zeta) + \eta_2 p_{22}(\zeta)], \quad (33)$$

where

$$\begin{aligned} p_{11}(\zeta) &= 25\zeta^2(5\zeta^2 - 7) - 130\sqrt{\eta_1}\zeta^3 + 4\eta_1(6\zeta^2 + 7), & p_{12}(\zeta) &= 8\zeta^2, \\ p_{21}(\zeta) &= 625\zeta^3(25\zeta^6 - 165\zeta^4 + 257\zeta^2 - 105) \\ &\quad - 250\eta_1^{1/2}\zeta^2(110\zeta^6 - 311\zeta^4 + 144\zeta^2 - 105) \\ &\quad + 100\eta_1\zeta(111\zeta^6 + 447\zeta^4 - 209\zeta^2 + 105) \\ &\quad - 40\eta_1^{3/2}(18\zeta^6 + 697\zeta^4 + 282\zeta^2 + 105) \\ &\quad + 96\eta_1^2\zeta(16\zeta^2 + 63), \\ p_{22}(\zeta) &= 8\zeta[25\zeta^2(23\zeta^4 - 73\zeta^2 + 3) - 10\sqrt{\eta_1}\zeta(27\zeta^4 + 67\zeta^2 - 18) + 12\eta_1(48\zeta^2 - 7)]. \end{aligned} \quad (34)$$

Below we divide the proof into three cases.

First case: $\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}_3 = \mathbf{0}$. In this case, $\eta_1 = \eta_2 = 0$, and

$$p(\lambda) = \frac{(\sqrt{\theta_{11}})^{13}}{125} \zeta^5 (5\zeta^2 - 7)^2 (5\zeta^4 - 26\zeta^2 + 15).$$

Obviously the roots of $p(\lambda)$ are all real. According to Lemma 2.3, we only need to prove $\hat{p}(\tilde{\mathbf{M}}_1) = \mathbf{0}$ for

$$\hat{p}(\lambda) = (\sqrt{\theta_{11}})^{13} \zeta (5\zeta^2 - 7) (5\zeta^4 - 26\zeta^2 + 15).$$

This can be verified by direct calculation.

Second case: $\mathbf{s}_1 = \mathbf{0}$ and $\mathbf{s}_2^2 + \mathbf{s}_3^2 > \mathbf{0}$. In this case, $\eta_1 = 0$, while the SPD property of $\boldsymbol{\Theta}$ gives $\eta_2 > 0$. The characteristic polynomial $p(\lambda)$ can be simplified as

$$p(\lambda) = \frac{\zeta^5}{78125} p_1(\zeta) p_2(\zeta),$$

where

$$\begin{aligned} p_1(\zeta) &= 25(5\zeta^2 - 7) + 8\eta_2, \\ p_2(\zeta) &= 25(5\zeta^2 - 7)(5\zeta^4 - 26\zeta^2 + 15) + 8\eta_2(23\zeta^4 - 73\zeta^2 + 3). \end{aligned}$$

When η_2 equals zero, all the roots of p_1 and p_2 are single and nonzero. Thus, when $\eta_2 < \delta$ for δ small enough, the roots of p_1 and p_2 are also single and nonzero. Furthermore, we claim that $p_1(\zeta)$ and $p_2(\zeta)$ have no common roots when δ is small enough. This can be proven following these steps:

1. Let $\tilde{p}_1(z) = p_1(\sqrt{z})$, $\tilde{p}_2(z) = p_2(\sqrt{z})$. Obviously, when δ is small enough, \tilde{p}_1 and \tilde{p}_2 are polynomials with all their roots positive. If \tilde{p}_1 and \tilde{p}_2 have no common roots, then p_1 and p_2 have no common roots.
2. The polynomial $\tilde{p}_1(z)$ is a linear function, and its only root is $(175 - 8\eta_2)/125$. The value of \tilde{p}_2 at this point is

$$\tilde{p}_2\left(\frac{175 - 8\eta_2}{125}\right) = \frac{8\eta_2(1152\eta_2^2 - 3400\eta_2 - 664375)}{15625}. \quad (35)$$

3. Since $0 < \eta_2 < \delta$, the value of (35) is negative if $\delta < (425 + 125\sqrt{3073})/288$, which means \tilde{p}_1 and \tilde{p}_2 have no common roots.

The above analysis shows when δ is small, $p_1(\zeta)p_2(\zeta)$ has no multiple roots, and $\zeta = 0$ is not a root of $p_1(\zeta)p_2(\zeta)$. Thus, the polynomial

$$q(\zeta) := \zeta p_1(\zeta)p_2(\zeta) \quad (36)$$

has no multiple roots if δ is small. Finally, it is verified by computer algebra system

$$q\left(\frac{\tilde{\mathbf{M}}_1 - u_1 \mathbf{I}}{\sqrt{\theta_{11}}}\right) = \mathbf{0}, \quad \text{if } s_1 = 0. \quad (37)$$

Hence, according the Lemma 2.3, the matrix $\tilde{\mathbf{M}}_1$ is diagonalizable.

Third case: $s_1 \neq \mathbf{0}$. In this case, $\eta_1 > 0$ and $\eta_2 > 0$. We first prove when δ is small, both $p_{11}(\zeta) + \eta_2 p_{12}(\zeta)$ and $p_{21}(\zeta) + \eta_2 p_{22}(\zeta)$ have no multiple or imaginary roots. When $\eta_1 = \eta_2 = 0$,

$$p_{11}(\zeta) + \eta_2 p_{12}(\zeta) = 25\zeta^2(5\zeta^2 - 7), \quad (38a)$$

$$p_{21}(\zeta) + \eta_2 p_{22}(\zeta) = 625\zeta^3(5\zeta^2 - 7)(5\zeta^4 - 26\zeta^2 + 15). \quad (38b)$$

Both polynomials have only real roots, and both of them have only one multiple roots — $\zeta = 0$. Thus, for small δ , if η_1 and η_2 are nonzero, then the multiple or imaginary roots must be around $\zeta = 0$ if they exist. For $p_{11}(\zeta) + \eta_2 p_{12}(\zeta)$, when δ is small, one can obtain its values at some particular points around $\zeta = 0$:

- $\zeta = -\sqrt{\eta_1}$: $p_{11}(-\sqrt{\eta_1}) + \eta_2 p_{12}(-\sqrt{\eta_1}) = 279\eta_1^2 - 147\eta_1 + 8\eta_1\eta_2 < 0$,
- $\zeta = 0$: $p_{11}(0) + \eta_2 p_{12}(0) = 28\eta_1 > 0$,
- $\zeta = \sqrt{\eta_1}$: $p_{11}(\sqrt{\eta_1}) + \eta_2 p_{12}(\sqrt{\eta_1}) = 19\eta_1^2 - 147\eta_1 + 8\eta_1\eta_2 < 0$.

This tells us that there are two distinct real roots of $p_{11}(\zeta) + \eta_2 p_{12}(\zeta)$ around $\zeta = 0$. Noting that $\zeta = 0$ is a root of multiplicity 2 of (38a), we conclude that in the case of $0 < \eta_1 < \delta$ and $0 < \eta_2 < \delta$, $p_{11}(\zeta) + \eta_2 p_{12}(\zeta)$ has no multiple or imaginary roots. Similarly, for $p_{21}(\zeta) + \eta_2 p_{22}(\zeta)$, when δ is small, one has

- $\zeta = -\sqrt{\eta_1}$:

$$\begin{aligned} p_{21}(-\sqrt{\eta_1}) + \eta_2 p_{22}(-\sqrt{\eta_1}) &= -\eta_1^{3/2}[54945\eta_1^3 - (106759 - 6760\eta_2)\eta_1^2 \\ &\quad + (193053 - 4632\eta_2)\eta_1 - (77175 + 1512\eta_2)] > 0, \end{aligned}$$

- $\zeta = 0$:

$$p_{21}(0) + \eta_2 p_{22}(0) = -4200\eta_1^{3/2} < 0,$$

- $\zeta = \frac{2}{5}\eta_1^{1/2} - \frac{4}{375}\eta_1^{3/2}$:

$$\begin{aligned} &p_{21}\left(\frac{2}{5}\eta_1^{1/2} - \frac{4}{375}\eta_1^{3/2}\right) + \eta_2 p_{22}\left(\frac{2}{5}\eta_1^{1/2} - \frac{4}{375}\eta_1^{3/2}\right) \\ &= \frac{64\eta_1^{7/2}}{9385585784912109375}[-4096\eta_1^{10} + 706560\eta_1^9 - 30182400\eta_1^8 \\ &\quad - 57600(29025 + 184\eta_2)\eta_1^7 + 4320000(48475 + 536\eta_2)\eta_1^6 \\ &\quad - 486000000(18575 + 428\eta_2)\eta_1^5 + 506250000(539275 + 19784\eta_2)\eta_1^4 \\ &\quad - 18984375000(412025 + 16176\eta_2)\eta_1^3 + 711914062500(209175 + 10576\eta_2)\eta_1^2 \\ &\quad - 40045166015625(26225 + 3704\eta_2)\eta_1 + 3003387451171875(175 + 484\eta_2)] > 0, \end{aligned}$$

- $\zeta = \sqrt{\eta_1}$:

$$p_{21}(\sqrt{\eta_1}) + \eta_2 p_{22}(\sqrt{\eta_1}) = -\eta_1^{3/2} [1495\eta_1^3 + (7019 - 2440\eta_2)\eta_1^2 - (98493 - 15352\eta_2)\eta_1 + (33075 + 1368\eta_2)] < 0.$$

This reveals that there are three distinct real roots of $p_{21}(\zeta) + \eta_2 p_{22}(\zeta)$ around $\zeta = 0$. Until now, the statement at the beginning of this paragraph has been proven.

The subsequent proof is divided into two parts:

1. If $\eta_1 = \eta_2$, then $p_{11}(\zeta) + \eta_2 p_{12}(\zeta)$ is a factor of $p_{21}(\zeta) + \eta_2 p_{22}(\zeta)$, and we actually have

$$p_{11}(\zeta) + \eta_2 p_{12}(\zeta) = (25\zeta^3 - 16\sqrt{\eta_1}\zeta^2 - 35\zeta - 14\sqrt{\eta_1})(5\zeta - 2\sqrt{\eta_1}), \quad (39)$$

$$p_{21}(\zeta) + \eta_2 p_{22}(\zeta) = [25\zeta(\zeta^4 - 26\zeta^2 + 15) + 30\sqrt{\eta_1}(3\zeta^4 + 6\zeta^2 + 5) - 192\eta_1\zeta] \times (25\zeta^3 - 16\sqrt{\eta_1}\zeta^2 - 35\zeta - 14\sqrt{\eta_1})(5\zeta - 2\sqrt{\eta_1}). \quad (40)$$

Thus we need to verify

$$p_{21} \left(\frac{\tilde{\mathbf{M}}_1 - u_1 \mathbf{I}}{\sqrt{\theta_{11}}} \right) + \eta_2 p_{22} \left(\frac{\tilde{\mathbf{M}}_1 - u_1 \mathbf{I}}{\sqrt{\theta_{11}}} \right) = \mathbf{0}. \quad (41)$$

According to Lemma 4.1, the condition $\eta_1 = \eta_2$ is equivalent to (28). Substitute (28) into the expression of $\tilde{\mathbf{M}}_1$, and (41) then can be directly verified.

2. If $\eta_1 \neq \eta_2$, the resultant of $p_{11} + \eta_2 p_{12}$ and $p_{21} + \eta_2 p_{22}$ is calculated as

$$\text{res}(p_{11} + \eta_2 p_{12}, p_{21} + \eta_2 p_{22}) = -100352000000000\eta_1^3(\eta_1 - \eta_2)^5 r(\eta_1, \eta_2), \quad (42)$$

where $r(\eta_1, \eta_2)$ is

$$\begin{aligned} r(\eta_1, \eta_2) = & 6519382474752\eta_1^5 + 7205633261568\eta_2\eta_1^4 - 1047028571136000\eta_1^4 + \\ & 2877437509632\eta_2^2\eta_1^3 + 71846341632000\eta_2\eta_1^3 + 6117273120960000\eta_1^3 + \\ & 488268103680\eta_2^3\eta_1^2 + 14075065958400\eta_2^2\eta_1^2 - 32261927040000\eta_2\eta_1^2 - \\ & 12991498038500000\eta_1^2 + 31436439552\eta_2^4\eta_1 + 74226585600\eta_2^3\eta_1 - \\ & 29723348160000\eta_2^2\eta_1 - 84800409000000\eta_2\eta_1 + 12363509395312500\eta_1 + \\ & 668860416\eta_2^5 - 13801881600\eta_2^4 - 707492160000\eta_2^3 + \\ & 13556709000000\eta_2^2 + 188918353125000\eta_2 - 3277351494140625. \end{aligned}$$

Evidently when δ is small, $r(\eta_1, \eta_2) < 0$. Noting that $\eta_1 > 0$ and $\eta_1 \neq \eta_2$, we conclude (42) is nonzero. According to Lemma 2.6, $p_{11}(\zeta) + \eta_2 p_{12}(\zeta)$ and $p_{21}(\zeta) + \eta_2 p_{22}(\zeta)$ have no common roots. Thus, the characteristic polynomial $p(\lambda)$ has no multiple roots, which gives us the diagonalizability of $\tilde{\mathbf{M}}_1$.

Final conclusion. For all the three cases listed above, it has been proven that when δ is small, all the eigenvalues of $\tilde{\mathbf{M}}_1$ are real, and the matrix $\tilde{\mathbf{M}}_1$ is diagonalizable. Thus the proof of Theorem 4.2 is completed. \square

Theorem 4.3. *There exists a positive constant $\delta > 0$, such that if $\rho^{-2} \mathbf{s}^T \Theta^{-1} \mathbf{s} < \delta$, the moment system (25) is hyperbolic.*

Proof. The hyperbolicity of the moment system (25) is equivalent to the diagonalizability of the matrix $n_k \tilde{\mathbf{M}}_k(\tilde{\mathbf{w}})$ for all unit vectors $\mathbf{n} = (n_1, n_2, n_3)^T \in \mathbb{R}^3$.

The rotational invariance of (25) implies that for any unit vector \mathbf{n} , there exists a constant square matrix \mathbf{R} such that

$$n_k \tilde{\mathbf{M}}_k(\tilde{\mathbf{w}}) = \mathbf{R}^{-1} \tilde{\mathbf{M}}_1(\mathbf{R}\tilde{\mathbf{w}})\mathbf{R}.$$

Actually, \mathbf{R} can be constructed as follows:

1. Construct an orthogonal matrix $\mathbf{G} = (g_{ij})_{3 \times 3}$ such that the first row of \mathbf{G} is (n_1, n_2, n_3) .
2. Define the “rotated moments” $\tilde{\mathbf{w}}'$ as

$$\tilde{\mathbf{w}}' = (\rho', u'_1, u'_2, u'_3, \theta'_{11}, \theta'_{22}, \theta'_{33}, \theta'_{12}, \theta'_{13}, \theta'_{23}, s'_1, s'_2, s'_3)^T,$$

where

$$\rho' = \rho, \quad u'_i = g_{ij}u_j, \quad \theta'_{ij} = g_{ik}g_{jl}\theta_{kl}, \quad s'_i = g_{ij}s_j. \quad (43)$$

3. The matrix \mathbf{R} is the unique matrix such that $\tilde{\mathbf{w}}' = \mathbf{R}\tilde{\mathbf{w}}$ for all $\tilde{\mathbf{w}}$.

According to Theorem 4.2, there exists a constant positive number δ such that the matrix $n_k \tilde{\mathbf{M}}_k(\tilde{\mathbf{w}})$ is diagonalizable if $\rho'^{-2} \mathbf{s}'^T (\boldsymbol{\Theta}')^{-1} \mathbf{s}' < \delta$, where $\boldsymbol{\Theta}' = (\theta'_{ij})_{3 \times 3}$. Using (43), we have

$$\rho'^{-2} \mathbf{s}'^T (\boldsymbol{\Theta}')^{-1} \mathbf{s}' = \rho^{-2} (\mathbf{G}\mathbf{s})^T (\mathbf{G}\boldsymbol{\Theta}\mathbf{G}^T)^{-1} (\mathbf{G}\mathbf{s}) = \rho^{-2} \mathbf{s}^T \boldsymbol{\Theta}^{-1} \mathbf{s}.$$

Thus the theorem is proven. \square

4.3. Quantification of the hyperbolicity region. The proof of Theorem 4.3 reveals that the maximal value of δ (denoted by δ_{\max} below) in Theorem 4.3 equals that in Theorem 4.2. Below we give a rough estimation of δ_{\max} . Let

$$\tilde{p} = (p_{11} + \eta_2 p_{12})(p_{21} + \eta_2 p_{22}),$$

where $p_{11}, p_{12}, p_{21}, p_{22}$ are defined in (34). We denote the domain on which the polynomial \tilde{p} has no imaginary roots to be Σ , and thus

$$\Sigma = \{(\eta_1, \eta_2) \mid \Im(\eta_1, \eta_2) = 0\},$$

where

$$\Im(\eta_1, \eta_2) := \max\{|\Im(z)| \mid z \text{ is the root of } \tilde{p}\}, \quad 0 \leq \eta_1 \leq \eta_2.$$

Since \Im is continuous, Σ has to be a closed region. We plot the domain Σ as the green area in Figure 1(a). The horizontal line $\eta_2 = \tilde{\delta}$ is tangent to the red curve. We have $\delta_{\max} \leq \tilde{\delta}$ and $\tilde{\delta} \approx 0.095$.

We denote the domain \mathcal{S} to be the domain on which the polynomial \tilde{p} has multiple roots. According to Lemma 2.4 and Lemma 2.6, we have that

$$\mathcal{S} = \{(\eta_1, \eta_2) \mid \Re(\eta_1, \eta_2) = 0\},$$

and

$$\Re(\eta_1, \eta_2) := \text{res}(\tilde{p}, \tilde{p}'), \quad \tilde{p}'(\zeta) = \frac{d}{d\zeta} \tilde{p}(\zeta), \quad 0 \leq \eta_1 \leq \eta_2.$$

Due to the continuity of the roots of polynomials with respect to its coefficients, we have $\partial\Sigma \subset \mathcal{S}$. Figure 1(b) shows part of \mathcal{S} . Comparing Figure 1(a) and Figure 1(b), we conclude that if $0 < \eta_1 < \eta_2 < \tilde{\delta}$, which implies that (η_1, η_2) is an interior point of Σ below the line $\eta_2 = \tilde{\delta}$, then $\tilde{\mathbf{M}}_1(\tilde{\mathbf{w}})$ is real diagonalizable.

In order to determine δ_{\max} , we have to consider two additional cases: (1) $\eta_1 = 0$, (2) $\eta_1 = \eta_2 > 0$. They correspond to the straight red lines in Figure 1. It can be argued as below for these cases:

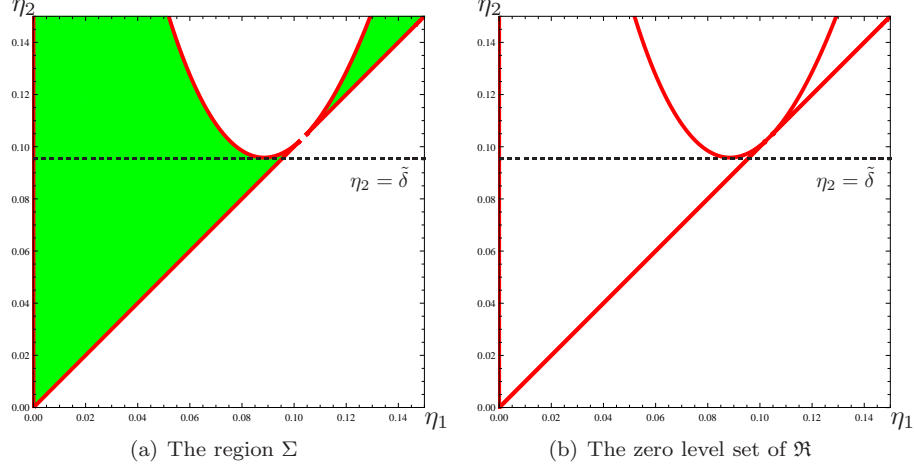


FIGURE 1. The x -axis stands for η_1 , and the y -axis stands for η_2

- For the case $\eta_1 = 0$, if $\eta_2 = 0$, the real diagonalizability of $\tilde{\mathbf{M}}_1$ has been proven. If $\eta_2 > 0$, since (37) always holds, we only need to consider whether the polynomial $q(\zeta)$ defined in (36) has multiple roots. Figure 2 gives the plots of $\text{res}(q, q')$ for $\eta_2 \in [0, 0.1]$, where $q'(\zeta) = \frac{d}{d\zeta}q(\zeta)$. It is found that if $0 < \eta_2 < \tilde{\delta} < 0.1$, then $\text{res}(q, q') > 0$, thus $q(\zeta)$ has no multiple roots. Then $\tilde{\mathbf{M}}_1$ is real diagonalizable.

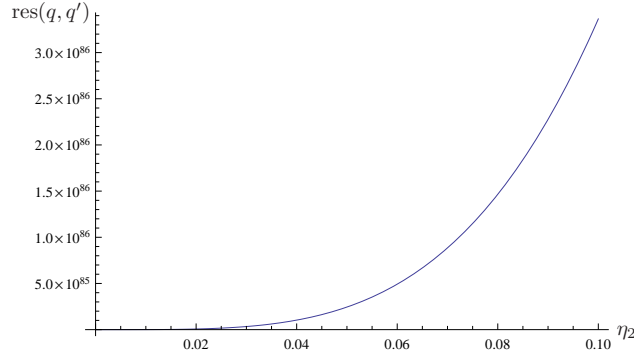
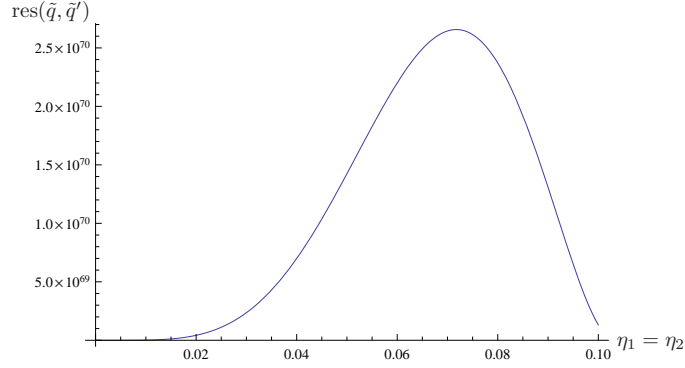


FIGURE 2. Plots of $\text{res}(q, q')$ in the case of $\eta_1 = 0$

- For the case $\eta_1 = \eta_2 > 0$, we have to study the multiplicities of the roots of (40). Denote the polynomial (40) by \tilde{q} , and let $\tilde{q}'(\zeta) = \frac{d}{d\zeta}\tilde{q}(\zeta)$. The values of $\text{res}(\tilde{q}, \tilde{q}')$ for $\eta_2 \in [0, 0.1]$ are given in Figure 3. It can also be observed that when $0 < \eta_1 = \eta_2 < \tilde{\delta} < 0.1$, $\text{res}(\tilde{q}, \tilde{q}')$ is greater than zero, which results in the real diagonalizability of $\tilde{\mathbf{M}}_1$.

As a summary, we claim that if $0 \leq \eta_1 \leq \eta_2 < \tilde{\delta}$, the moment system (25) is hyperbolic. Thus $\delta_{\max} = \tilde{\delta} \approx 0.095$.

In order to give a more precise description of the size of the hyperbolicity region, we apply the Chapman-Enskog method to the modified 13-moment system (26).

FIGURE 3. Plots of $\text{res}(\tilde{q}, \tilde{q}')$ in the case of $\eta_1 = \eta_2$

Apply the transformation $t' = \varepsilon t$ and $\mathbf{x}' = \varepsilon \mathbf{x}$ to (26), and then the moment system becomes

$$\frac{\partial \tilde{\mathbf{w}}}{\partial t'} + \tilde{\mathbf{M}}_k(\tilde{\mathbf{w}}) \frac{\partial \tilde{\mathbf{w}}}{\partial x'_k} = \frac{1}{\varepsilon} \tilde{\mathbf{Q}}(\tilde{\mathbf{w}}). \quad (44)$$

For small ε , we formally expand $\tilde{\mathbf{w}}$ as

$$\tilde{\mathbf{w}} = \tilde{\mathbf{w}}^{(0)} + \varepsilon \tilde{\mathbf{w}}^{(1)} + \varepsilon^2 \tilde{\mathbf{w}}^{(2)} + \dots$$

The Chapman-Enskog expansion fixes the leading order term $\tilde{\mathbf{w}}^{(0)}$ to be the equilibrium part of $\tilde{\mathbf{w}}$:

$$\rho = \rho^{(0)}, \quad \mathbf{u} = \mathbf{u}^{(0)}, \quad \boldsymbol{\Theta} = \theta \mathbf{I} + \varepsilon \boldsymbol{\Theta}^{(1)} + \varepsilon^2 \boldsymbol{\Theta}^{(2)} + \dots, \quad \mathbf{s} = \varepsilon \mathbf{s}^{(1)} + \varepsilon^2 \mathbf{s}^{(2)} + \dots \quad (45)$$

Substituting (45) into (44) and balancing the zeroth order terms on both sides of (44), one may conclude

$$\theta_{ij}^{(1)} = -\frac{2\mu}{\rho} \left(\frac{\partial v_{(i}}{\partial x'_{j)}} - \frac{1}{3} \frac{\partial v_k}{\partial x'_k} \right), \quad s_j^{(1)} = -\frac{15\mu}{4\theta} \frac{\partial \theta}{\partial x'_i}.$$

These are equivalent to the well-known Navier-Stokes and Fourier laws.

According to the expansion (45), we have

$$\boldsymbol{\Theta}^{-1} = \theta^{-1} \mathbf{I} - \frac{\varepsilon}{\theta^2} \boldsymbol{\Theta}^{(1)} + O(\varepsilon^2),$$

and thus

$$\rho^{-2} \mathbf{s}^T \boldsymbol{\Theta}^{-1} \mathbf{s} = \varepsilon^2 \rho^{-2} \theta^{-1} |\mathbf{s}^{(1)}|^2 + O(\varepsilon^3) = \frac{225}{16} \varepsilon^2 \mu^2 \rho^{-2} \theta^{-3} |\nabla_{\mathbf{x}'} \theta|^2 + O(\varepsilon^3). \quad (46)$$

For Maxwellian molecules, the viscosity μ can be related to the mean free path l_{mfp} by

$$\mu = \rho l_{\text{mfp}} \sqrt{\frac{\pi \theta}{2}}.$$

Thus, (46) is simplified as

$$\rho^{-2} \mathbf{s}^T \boldsymbol{\Theta}^{-1} \mathbf{s} = \frac{225\pi}{32} \varepsilon^2 \left(\frac{|\nabla_{\mathbf{x}'} \theta|}{\theta} l_{\text{mfp}} \right)^2 + O(\varepsilon^3).$$

Neglecting the high order terms, we get

$$\rho^{-2} \mathbf{s}^T \boldsymbol{\Theta}^{-1} \mathbf{s} \approx \frac{225\pi}{32} \varepsilon^2 \left(\frac{|\nabla_{\mathbf{x}'} \theta|}{\theta} l_{\text{mfp}} \right)^2 = \frac{225\pi}{32} \left(\frac{|\nabla_{\mathbf{x}} \theta|}{\theta} l_{\text{mfp}} \right)^2. \quad (47)$$

Thus the hyperbolicity condition $\rho^{-2} \mathbf{s}^T \boldsymbol{\Theta}^{-1} \mathbf{s} < \delta_{\max}$ is approximately given as

$$|\nabla_{\mathbf{x}} \theta| < C_{\text{hyp}} \theta / l_{\text{mfp}}, \quad C_{\text{hyp}} = \sqrt{\frac{32\delta_{\max}}{225\pi}}.$$

Since $\delta_{\max} \approx 0.095$, we have that $C_{\text{hyp}} \approx 0.065$. Thus the temperature is allowed to change around 6.5% of its value in one mean free path in order to ensure the hyperbolicity. Consider the symmetric plane Couette flow problem. The Navier-Stokes equations together with the first-order slip boundary condition is valid only for $l_{\text{mfp}} \leq 0.1L$, where L is the distance between plates [7]. For $Kn = l_{\text{mfp}}/L$, in order to satisfy the criterion (47), the ratio of the temperature in the middle of the two plates to the temperature on each plate must not exceed $Kn^{-1}C_{\text{hyp}}$. The numerical results in [8] show that such a criterion is satisfied even for very fast plate velocities.

5. Conclusion. We find that for Grad's 13-moment system, the equilibrium is always on the boundary of its hyperbolicity region. A modified 13-moment system is proposed so that the local hyperbolicity around the equilibrium states can be achieved. The derivation of this new model is almost the same as the original one, except that the basis functions used in the expansions of the distribution functions are different. Obviously, this new model is far away from perfection; most of the classical criticism on Grad's 13-moment system still applies to this new model. However, due to the similarity of these two systems, the techniques developed for Grad's 13-moment system may also apply to this new model. This modified system enriches the 13-moment family, and some interesting aspects are found for this new member.

REFERENCES

- [1] G. A. Bird. *Molecular Gas Dynamics and the Direct Simulation of Gas Flows*. Oxford: Clarendon Press, 1994.
- [2] S. Chapman and T. G. Cowling. *The Mathematical Theory of Non-uniform Gases, Third Edition*. Cambridge University Press, 1990.
- [3] H. Grad. Note on N -dimensional Hermite polynomials. *Comm. Pure Appl. Math.*, 2(4):325–330, 1949.
- [4] H. Grad. On the kinetic theory of rarefied gases. *Comm. Pure Appl. Math.*, 2(4):331–407, 1949.
- [5] S. Jin, L. Pareschi, and M. Slemrod. A relaxation scheme for solving the Boltzmann equation based on the Chapman-Enskog expansion. *Acta Math. Appl. Sin.-E.*, 18(1):37–62, 2002.
- [6] S. Jin and M. Slemrod. Regularization of the Burnett equations via relaxation. *J. Stat. Phys.*, 103(5–6):1009–1033, 2001.
- [7] G. Karniadakis, A. Beskok, and N. Aluru. *Microflows and Nanoflows: Fundamentals and Simulation*, volume 29 of *Interdisciplinary Applied Mathematics*. Springer, New York, U.S.A., 2005.
- [8] L. Mieussens and H. Struchtrup. Numerical comparison of Bhatnagar-Gross-Krook models with proper Prandtl number. *Phys. Fluids*, 16(8):2797–2813, 2004.
- [9] I. Müller and T. Ruggeri. *Rational Extended Thermodynamics, Second Edition*, volume 37 of *Springer tracts in natural philosophy*. Springer-Verlag, New York, 1998.
- [10] Wolfram Research. *Mathematica 9*. <http://www.wolfram.com/mathematica>.
- [11] H. Struchtrup. Derivation of 13 moment equations for rarefied gas flow to second order accuracy for arbitrary interaction potentials. *Multiscale Model. Simul.*, 3(1):221–243, 2005.
- [12] H. Struchtrup and M. Torrilhon. Regularization of Grad's 13 moment equations: Derivation and linear analysis. *Phys. Fluids*, 15(9):2668–2680, 2003.
- [13] M. Torrilhon. Regularized 13-moment-equations. In M. S. Ivanov and A. K. Rebrov, editors, *Rarefied Gas Dynamics: 25th International Symposium*, 2006.

- [14] M. Torrilhon. Hyperbolic moment equations in kinetic gas theory based on multi-variate Pearson-IV-distributions. *Commun. Comput. Phys.*, 7(4):639–673, 2010.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: caizn@pku.edu.cn

E-mail address: ywfan@pku.edu.cn

E-mail address: rli@math.pku.edu.cn